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Existence of stationary geodesics of left-invariant Lagrangians

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Abstract

Stationary geodesics of left-invariant Riemannian metrics on Lie groups were introduced by Arnold as those geodesics which are also orbits of one-parameter groups of left-translations. The existence of infinitely many stationary geodesics in the case of compact semi-simple Lie groups has recently been established by Szenthe. Stationary geodesics of left-invariant Lagrangians on Lie groups are studied and the existence of infinitely many such geodesics on compact semi-simple Lie groups is established below.

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Geodesics of left-invariant Riemannian metrics on Lie groups were studied by Arnold extending Euler's theory of rigid-body motion [A]. A major part of Arnold's paper is devoted to the study of stationary geodesics; these are those geodesics which are simultaneously orbits of one-parameter groups of left-translations. A basic fact concerning stationary geodesics is the existence of a correspondence which renders to each such geodesic a critical point of that function which is obtained from the energy function of the left-invariant Riemannian metric by restricting it to an adjoint orbit. The above correspondence has a basic role in Arnold's stability theory of stationary geodesics.

Being unaware of Arnold's paper, stationary geodesics of left-invariant Riemannian metrics of Lie groups were studied by the author under the name of homogeneous geodesics [Sz]. The correspondence of stationary geodesics to critical points of the restricted energy function was rediscovered and proved within the framework of Riemannian geometry in [Sz]; actually Arnold's proof of the same fact was motivated by methods of mechanics. The main result of the paper [Sz] concerns the existence of stationary geodesics. It was shown that on a compact semi-simple Lie group of rank $\geqslant 2$ there are infinitely many stationary geodesics in the case of any left-invariant Riemannian metric.

The above-mentioned existence theorem of stationary geodesics of left-invariant Riemannian metrics is generalized to stationary geodesics of left-invariant Lagrangians in the present paper. As a starting point a correspondence of stationary geodesics to critical points of restricted Lagrangians is established under the assumption that the Lie group is compact and Lagrangian is a first integral of its Lagrangian field. Then results concerning the existence of stationary geodesics are obtained. First, the existence of at least two stationary geodesics is proved in the case of any compact Lie group and left-invariant Lagrangian which is a first integral of its Lagrangian field. Secondly, the existence of infinitely many stationary geodesics is established in the case of any compact semi-simple Lie group of rank $\geqslant 2$ and left-invariant Lagrangian which is a first integral of its Lagrangian field.

1. Left-invariant Lagrangians over Lie groups and their stationary geodesics

Some basic concepts and facts of Lagrangian analytical mechanics are summarized in what follows.

Let M be a smooth manifold, a continuous function $L : TM \to \mathbb{R}$ is said to be a *Lagrangian* over M if it is smooth on the open set $TM^\circ = TM - \{\mathcal{O}_{TM}\}$, where $\mathcal{O}_{TM}$ is the image of the zero section of TM. The *vertical endomorphism* υ of the second tangent bundle TTM yields the *vertical derivation* ι_υ and by $\mathrm{d}_\upsilon = \iota_\upsilon \mathrm{d} - \mathrm{d}\iota_\upsilon$ the *vertical differential* (see, e.g., [G, pp 159–64]). Therefore, the vertical differential $\mathrm{d}_\upsilon L$ of L, a differential 1-form on TM°, is obtained (see, e.g., [G, pp 161–4]). The Lagrangian L is said to be *regular* if the differential 2-form $\mathrm{dd}_\upsilon L$ is non-degenerate. The *Liouville field* $A : TM \to TTM$ is the smooth vector field induced by the one-parameter dilatation group of TM as a vector bundle (see, e.g., [G, pp 155–6]). A smooth function $F : TM^\circ \to \mathbb{R}$ is said to be *homogeneous of degree* k if $AF = kF$ holds and a smooth vector field $Z : TM \to T(TM^\circ)$ is said to be *homogeneous of degree* k if $[A, Z] = (k-1)Z$ is valid where $k \in \mathbb{R}$. Now the *Euler–Lagrange equation* for L is obtainable in the following form:

$$\iota_X \mathrm{dd}_\upsilon L = \mathrm{d}(L - AL)$$

where a solution of the equation is a smooth vector field $X : TM^\circ \to T(TM^\circ)$ satisfying the above equality and it is called a *Lagrangian field* associated with L (see, e.g., [B, pp 22–6]). If L is regular then such an X uniquely exists. Let L be regular, if $\hat{\gamma} : I \to TM^\circ$ is a maximal integral curve of the Lagrangian field X then $\gamma = \pi_M \circ \hat{\gamma} : I \to M$ is called a *geodesic* of the Lagrangian L. In this case $\hat{\gamma} = \dot{\gamma}$ is valid (see, e.g., [G, pp 169–75] and [B, pp 22–9]). A smooth vector field $S : TM \to TTM$ is called an *infinitesimal symmetry of the Lagrangian* L if $SL = 0$ holds and S is said to be an *infinitesimal symmetry of the Lagrangian field* X if $[S, X] = 0$ is valid.

Proposition 1.1. *Let $L : TM \to \mathbb{R}$ be a regular Lagrangian over a smooth manifold M and $S : TM \to TTM$ be an infinitesimal symmetry of L which is homogeneous of degree one and such that $\mathcal{L}_S\upsilon = 0$ holds for the vertical endomorphism υ. Then S is an infinitesimal symmetry of the Lagrangian field X of L.*

Proof. Let $Z : TM \to TTM$ be a smooth vector field and υ the vertical endomorphism of TTM, then by $\upsilon' = \mathcal{L}_Z\upsilon$ an endomorphism υ' of TTM is obtained. Consider the corresponding derivation $\iota_{\upsilon'}$ and the differential $\mathrm{d}_{\upsilon'} = [\iota_{\upsilon'}, \mathrm{d}]$. Then the following fundamental commutator identity is valid:

$$[\mathcal{L}_Z, \mathrm{d}_\upsilon] = \mathrm{d}_{\upsilon'}.$$

In fact, the validity of the above identity can be verified by checking it in the case of the differential forms ϕ, $\mathrm{d}\phi$, $\phi \in \mathcal{F}(TM)$. Assume that the smooth vector field S is an infinitesimal symmetry of the Lagrangian L and $\mathcal{L}_S\upsilon = 0$ is valid. Then by the Euler–Lagrange equation and the above commutator identity the following holds:

$$\begin{aligned}0 &= \iota_X\, \mathrm{dd}_\upsilon SL = \iota_X\, \mathrm{dd}_\upsilon \mathcal{L}_S L\\ &= \iota_X\, \mathrm{d}(\mathrm{d}_\upsilon \mathcal{L}_S - \mathcal{L}_S\, \mathrm{d}_\upsilon)L + \iota_X\, \mathrm{d}\mathcal{L}_S\, \mathrm{d}_\upsilon L\\ &= -\iota_X\, \mathrm{dd}_{\upsilon'} L + \iota_X \mathcal{L}_S\, \mathrm{dd}_\upsilon L\\ &= (\iota_X \mathcal{L}_S - \mathcal{L}_S \iota_X)\, \mathrm{dd}_\upsilon L + \mathcal{L}_S \iota_X\, \mathrm{dd}_\upsilon L\\ &= \iota_{[X,S]}\, \mathrm{dd}_\upsilon L + \mathcal{L}_S\, \mathrm{d}(L - AL) = \iota_{[X,S]}\, \mathrm{dd}_\upsilon L + \mathrm{d}(SL + [A, S]L - ASL)\\ &= \iota_{[X,S]}\, \mathrm{dd}_\upsilon L\end{aligned}$$

where another well known basic commutator identity and the assumption that S is homogeneous of degree one have been applied (see, e.g., [G]). However, then as L is regular $[X, S] = 0$ follows. □

Concerning Lie groups the following elementary facts will be applied in subsequent definitions:

Let G be a connected Lie group, $\lambda : G \times G \to G$ the action being defined by the left-translations $\lambda_g : G \to G$, $g \in G$ and $T\lambda : G \times TG \to TG$ the action given by the tangent linear maps $T\lambda_g : TG \to TG$, $g \in G$ of the left-translations. The infinitesimal generators of the action λ are the right-invariant vector fields; namely, if $Z \in \boldsymbol{g}$ is an element of the Lie algebra of G and $\bar{Z} : G \to TG$ is the corresponding infinitesimal generator of λ then

$$\begin{aligned}\bar{Z}(x) &= \frac{\mathrm{d}}{\mathrm{d}\tau}(\lambda_{\mathrm{Exp}(\tau Z)}x)\bigg|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\rho_x\, \mathrm{Exp}(\tau Z))\bigg|_{\tau=0}\\ &= T_e\rho_x \bar{Z}(e) \qquad x \in G\end{aligned}$$

holds where $\rho_x : G \to G$ is the right-translation by x. Now $\hat{Z} : TG \to TTG$ the infinitesimal generator of the action $T\lambda$ corresponding to Z is obtainable also as the complete lift $\bar{Z}^c$ of the right-invariant field $\bar{Z}$, in other words $\hat{Z} = \bar{Z}^c$ holds, according to a basic result (see, e.g., [MFVMR, pp 156–8]).

Definition. Let G be a connected Lie group, a regular Lagrangian $L : TG \to \mathbb{R}$ is said to be *left-invariant* if it is invariant under the action $T\lambda$. Moreover, a smooth vector field $X : TG^\circ \to TTG$ is said to be *left-invariant* if

$$TT\lambda_g \circ X \circ T\lambda_g^{-1} = X \qquad g \in G$$

holds, where $TG^\circ = TG - \mathcal{O}_{TG}$ as before.

The preceding proposition obviously has the following:

Corollary. *If $L : TG \to \mathbb{R}$ is a left-invariant Lagrangian then its Lagrangian field X is left-invariant as well.*

Definition. *A geodesic $\gamma : I \to G$ of a left-invariant Lagrangian is said to be* stationary *if there is a $Z \in \boldsymbol{g}$ such that $\gamma(\tau) = \lambda_{\mathrm{Exp}(\tau Z)}\gamma(0)$, $\tau \in I$ holds. It will be said that $v \in T_eG$ is a* geodesic vector *if the geodesic $\gamma : I \to G$ defined by $v = \dot{\gamma}(0)$ is stationary.*

Lemma 1.2. *Let L be a left-invariant Lagrangian over a connected Lie group G with its Lagrangian field $X : TG^\circ \to T(TG^\circ)$. If $\hat{\gamma} : I \to TG^\circ$ is a maximal integral curve of X then the geodesic $\gamma = \pi_G \circ \hat{\gamma} : I \to G$ is stationary if and only if there is a $Z \in \boldsymbol{g}$ such that $\hat{\gamma}(\tau) = T\lambda_{\mathrm{Exp}(\tau Z)}\hat{\gamma}(0)$, $\tau \in I$ holds.*

Proof. Assume first that the geodesic γ is stationary. Then there is a $Z \in \boldsymbol{g}$ such that for a $\tau_0 \in I$ the following holds:

$$\begin{aligned}\hat{\gamma}(\tau_0) = \dot{\gamma}(\tau_0) &= \frac{\mathrm{d}}{\mathrm{d}\tau}(\gamma(\tau_0+\tau))\Big|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}(\lambda_{\mathrm{Exp}((\tau_0+\tau)Z)}\gamma(0))\Big|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\lambda_{\mathrm{Exp}(\tau_0 Z)}\gamma(\tau))\Big|_{\tau=0} \\ &= T\lambda_{\mathrm{Exp}(\tau_0 Z)}\dot{\gamma}(0) = T\lambda_{\mathrm{Exp}(\tau_0 Z)}\hat{\gamma}(0).\end{aligned}$$

Assume now conversely that for the maximal integral curve $\hat{\gamma}$ of X the following holds $\hat{\gamma}(\tau) = T\lambda_{\mathrm{Exp}(\tau Z)}\hat{\gamma}(0)$, $\tau \in I$ with some $Z \in \boldsymbol{g}$. Then the following is also valid:

$$\begin{aligned}\gamma(\tau) &= \pi_G(\hat{\gamma}(\tau)) = \pi_G(T\lambda_{\mathrm{Exp}(\tau Z)}\hat{\gamma}(0)) \\ &= \lambda_{\mathrm{Exp}(\tau Z)}(\pi_G(\hat{\gamma}(0)) = \lambda_{\mathrm{Exp}(\tau Z)}\gamma(0)\end{aligned}$$

which shows that γ is a stationary geodesic. □

Proposition 1.3. *Let G be a connected Lie group, $L : TG \to \mathbb{R}$ a left-invariant Lagrangian, X its Lagrangian field and $v \in T_eG - \{0_e\}$. Then v is a geodesic vector if and only if there is a $Z \in \boldsymbol{g}$ such that $X(v) = \bar{Z}^c(v)$ holds with the complete lift of the infinitesimal generator of λ corresponding to Z.*

Proof. The curve $\tau \mapsto T_e\lambda_{\mathrm{Exp}(\tau Z)}(v)$, $\tau \in \mathbb{R}$ is obviously an integral curve of the field $\hat{Z} = \bar{Z}^c$. As $[A, \bar{Z}^c] = 0$ holds by a basic observation (see, e.g., [MFVMR, pp 156–8]) and as $\mathcal{L}_{\bar{Z}^c} v = 0$ is also valid by a fundamental result (see, e.g., [MFVMR, pp 160–1]), the former proposition 1.1 applies and yields that $\bar{Z}^c$ is an infinitesimal symmetry of X. Therefore,

$$TT\lambda_{\mathrm{Exp}(\tau Z)} \circ X \circ T\lambda^{-1}_{\mathrm{Exp}(\tau Z)}(v) = X(v)$$

is valid (see, e.g., [G, pp 104–7]). However, then the following is obtained:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(T_e\lambda_{\mathrm{Exp}(\tau Z)}(v))\Bigg|_{\tau=\tau_0} &= \bar{Z}^c \circ T_e\lambda_{\mathrm{Exp}(\tau_0 Z)}(v) \\ &= TT\lambda_{\mathrm{Exp}(\tau_0 Z)} \circ \bar{Z}^c \circ T\lambda^{-1}_{\mathrm{Exp}(\tau_0 Z)} \circ T\lambda_{\mathrm{Exp}(\tau_0 Z)}(v) \\ &= TT\lambda_{\mathrm{Exp}(\tau_0 Z)} \circ \bar{Z}^c(v) = TT\lambda_{\mathrm{Exp}(\tau_0 Z)} \circ X(v) \\ &= TT\lambda_{\mathrm{Exp}(\tau_0 Z)} \circ X \circ T\lambda^{-1}_{\mathrm{Exp}(\tau_0 Z)} \circ T\lambda_{\mathrm{Exp}(\tau_0 Z)}(v) \\ &= X \circ T\lambda_{\mathrm{Exp}(\tau_0 Z)}(v).\end{aligned}$$

The above equalities yield that the curve $\tau \mapsto T\lambda_{\mathrm{Exp}(\tau Z)}(v)$, $\tau \in \mathbb{R}$ is an integral curve of the field X as well; but then this curve is equal to the curve $\hat{\gamma}$. Therefore, $\gamma = \pi_G \circ \hat{\gamma}$ is a stationary geodesic of L by lemma 1.1. □

As some of the subsequent calculations are based on special coordinate systems a concise account of their construction and of some basic properties is presented in what follows.

Consider first a *canonical coordinate system* of the Lie group G defined on a neighbourhood $U \subset G$ of e as follows (see also, e.g., [C, pp 115–8]). Let T_eG be canonically identified with $\boldsymbol{g}$ as a vector space and consider an open neighbourhood $U' \subset T_eG$ of 0_e such that $\mathrm{Exp}\lceil U'$ is a diffeomorphism and put $U = \mathrm{Exp}(U')$. Fix a base $(E_1, \ldots, E_m)$ of T_eG and let $\omega : T_eG \to \mathbb{R}^m$ be the vector space isomorphism defined by $\omega : T_eG \ni \sum_{i=1}^m \lambda^i E_i \mapsto (\lambda^1, \ldots, \lambda^m) \in \mathbb{R}^m$. Now put

$$\omega \circ (\mathrm{Exp}\lceil U)^{-1}(g) = (y^1(g), \ldots, y^m(g)) \in \mathbb{R}^m \qquad g \in U.$$

Consider now $Z = \sum_{j=1}^{m} \zeta^j E_j \in \boldsymbol{g}$ then the corresponding one-parameter group of left-translations $\lambda_{\mathrm{Exp}(\tau Z)} : G \to G, \tau \in \mathbb{R}$ yields the infinitesimal generator $\bar{Z}$ of the action λ which can be obtained in the above canonical coordinate system as follows taking into account the Campbell–Hausdorff formula as well:

$$
\begin{aligned}
\bar{Z}(g) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{Exp}\left(\tau \sum_{i=1}^{m} \zeta^i E_i\right) \mathrm{Exp}\left(\sum_{j=1}^{m} y^j E_j\right)\right)\Bigg|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{Exp}\left(\sum_{i=1}^{m} (\tau\zeta^i + y^i) E_i + \tau \sum_{i,j=1}^{m} y^i \zeta^j [E_i, E_j] + \tau^2 R\right)\right)\Bigg|_{\tau=0} \\
&= T_X \,\mathrm{Exp} \circ \kappa_X \left(\sum_{k=1}^{m} \zeta^k E_k + \sum_{i,j=1}^{m} c_{ij}^k y^i \zeta^j E_k\right) \\
&= \sum_{k=1}^{m} \left(\zeta^k + \sum_{ij=1}^{m} c_{ij}^k y^i \zeta^j\right) \frac{\partial}{\partial y^k}\bigg|_g
\end{aligned}
$$

where $U \ni g = \mathrm{Exp}(X), X \in U'$ and R is the remainder term of the Campbell–Hausdorff formula (see, e.g., [T, pp 119–32]).

Secondly, following the general construction by which a coordinate system of a smooth manifold induces a natural coordinate system of its tangent bundle, also the canonical coordinate system $(y^1, \ldots, y^m)$ of G induces a *natural coordinate system*

$$(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m)$$

of the tangent bundle TG on its open subset TU as follows. If $w = \sum_{i=1}^{m} w^i \frac{\partial}{\partial y^i}\big|_g \in T_g G$ for $g \in U$ then $x^i(w) = y^i \circ \pi_G(w), \dot{x}^i(w) = w^i, i = 1, \ldots, m$.

Now the expression of the complete lift $\bar{Z}^c$ of the infinitesimal generator $\bar{Z}$ of the action $\lambda : G \times G \to G$ in the above natural coordinate system of TG is obtained as follows:

$$\bar{Z}^c \lceil TU = \sum_{k=1}^{m} \left\{ \left(\zeta^k + \sum_{ij=1}^{m} c_{ij}^k \zeta^i x^j\right) \frac{\partial}{\partial x^k} + \sum_{ij=1}^{m} c_{ij}^k \zeta^i \dot{x}^j \frac{\partial}{\partial \dot{x}^k} \right\}$$

in fact, a direct application of the general coordinate expression of complete lifts of vector fields yields the above one (see, e.g., [MFVMR, pp 156–8]).

Proposition 1.4. *Let $L : TG \to \mathbb{R}$ be a left-invariant Lagrangian over a connected Lie group, X its Lagrangian field, $v \in T_eG - \{0_e\}$ and $\gamma : I \to G$ the maximal geodesic defined by $\dot{\gamma}(0) = v$. Let $(y^1, \ldots, y^m)$ be a canonical coordinate system of G as a Lie group defined by the base $(E_1, \ldots, E_m)$ of T_eG and $(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m)$ the induced natural coordinate system of TG on TU and*

$$X \lceil TU = \sum_{k=1}^{m} \left\{ \dot{x}^k \frac{\partial}{\partial x^k} + \xi^k(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m) \frac{\partial}{\partial \dot{x}^k} \right\}.$$

Then v is a geodesic vector if and only if $\xi^k(0, \ldots, 0; v^1, \ldots, v^m) = 0, k = 1, \ldots, m$ holds where v has the coordinates $(0, \ldots, 0; v^1, \ldots, v^m)$ in the above induced natural coordinate system of TG.

Proof. According to proposition 1.3 the geodesic γ is stationary if and only if with some $Z \in \boldsymbol{g}$ the following holds:

$$\sum_{k=1}^{m}\left\{\zeta^k \frac{\partial}{\partial x^k} + \sum_{i,j=1}^{m} c_{ij}^k \zeta^i v^j \frac{\partial}{\partial \dot{x}^k}\right\} = \bar{Z}^c(v) = X(v)$$

$$= \sum_{l=1}^{m}\left\{v^l \frac{\partial}{\partial x^l} + \xi^l(0,\ldots,0;v^1,\ldots,v^m)\frac{\partial}{\partial \dot{x}^l}\right\}$$

where the above derived coordinate expression of $\bar{Z}^c$ has been applied. However, the above equality holds if and only if the following is valid:

$$\zeta^k = v^k \qquad \sum_{i,j=1}^{m} c_{ij}^k \zeta^i v^j = \xi^k(0,\ldots,0;v^1,\ldots,v^m) \qquad k = 1,\ldots,m.$$

However, if the above equalities are valid then $c_{ij}^k = -c_{ji}^k$, $i, j, k = 1,\ldots,m$ implies that

$$\xi^k(0,\ldots,0;v^1,\ldots,v^m) = 0 \qquad k = 1,\ldots,m$$

follows. Assume now conversely that $\xi^k(0,\ldots,0;v^1,\ldots,v^m) = 0$, $k = 1,\ldots,m$ is valid for some $v \in T_eG - \{0\}$. Consider now $Z = \sum_{i=1}^m v^i E_i \in \boldsymbol{g}$. Then $\bar{Z}^c(v) = X(v)$ is valid by the expressions above. □

2. Stationary geodesics and the critical points of the restricted Lagrangian

Some elementary facts concerning adjoint actions are summarized in what follows.

Let $ad : G \times G \to G$ be the adjoint action given by $ad(g)x = gxg^{-1}$, $(g, x) \in G \times G$ and $Ad : G \times TG \to TG$ the induced action defined by

$$Ad(g)v = T_x ad(g)v \qquad v \in T_xG \quad x \in G.$$

In order to calculate the infinitesimal generators of the action ad, fix $Z \in \boldsymbol{g} = T_eG$ and $x = \mathrm{Exp}(W)$, $W \in \boldsymbol{g}$; then by the equivariance of Exp the following holds for the infinitesimal generator $\tilde{Z}$ of the action ad induced by Z:

$$\tilde{Z}(x) = \frac{\mathrm{d}}{\mathrm{d}\tau}(ad(\mathrm{Exp}(\tau Z))x)\Big|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}(ad(\mathrm{Exp}(\tau Z))\,\mathrm{Exp}\,W)\Big|_{\tau=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{Exp}(Ad(\mathrm{Exp}(\tau Z)W))\Big|_{\tau=0} = T_W\,\mathrm{Exp}\circ\kappa_W([Z, W])$$

where $\boldsymbol{g}$ as a vector space is canonically identified with T_eG and $\kappa_W : T_eG \to T_W T_eG$ is the canonical isomorphism (see, e.g., [T, pp 98–9]). Now let $(y^1,\ldots,y^m)$ be the canonical coordinate system of the Lie group defined on a neighbourhood U of e by a base $(E_1,\ldots,E_m)$ of T_eG. Then $Z = \sum_{i=1}^m \zeta^i E_i$, $W = \sum_{j=1}^m v^j E_j$ and the following is obtained:

$$\tilde{Z}(x) = T_W\,\mathrm{Exp}\circ\kappa_W\left(\left[\sum_{i=1}^m \zeta^i E_i, \sum_{j=1}^m y^j E_j\right]\right) = T_W\,\mathrm{Exp}\circ\kappa_V\left(\sum_{i,j,k=1}^m c_{ij}^k \zeta^i v^j E_k\right)$$

$$= \sum_{i,j,k=1}^m c_{ij}^k \zeta^i v^j T_W\,\mathrm{Exp}\circ\kappa_W(E_k) = \sum_{ij=1}^m c_{ij}^k \zeta^i y^j \frac{\partial}{\partial y^k}\Big|_g$$

where c_{ij}^k, $i, j, k = 1,\ldots,m$ are the structural constants of $\boldsymbol{g}$ with respect to the above fixed base $(E_1,\ldots,E_m)$. Now let $\tilde{Z}^c$ be the complete lift of $\tilde{Z}$ to TG. Then the expression of $\tilde{Z}^c$ in the natural coordinate system $(x^1, \dots, x^m; \dot{x}^1, \dots \dot{x}^m)$ of TG induced by $(y^1, \dots, y^m)$ is the following:

$$\tilde{Z}^c \lceil TU = \sum_{k=1}^{m} \left\{ \sum_{i,j=1}^{m} \left(c_{ij}^k \zeta^i x^j \frac{\partial}{\partial x^k} + \sum_{i,j=1}^{m} c_{ij}^k \zeta^i \dot{x}^j \frac{\partial}{\partial \dot{x}^k} \right) \right\}$$

according to a basic expression of complete lifts of vector fields (see, e.g., [MFVMR, pp 156–8]). □

Definition. Let G be a connected Lie group, $L : TG \to \mathbb{R}$ a left-invariant Lagrangian over G and $v \in T_eG - \{0_e\}$. Then $G(v) \subset T_eG$ the orbit of v under the adjoint action $Ad : G \times TG \to TG$ is obtainable as the image of the canonical equivariant injective immersion

$$\chi : G/G_v \to G(v) \subset T_eG.$$

Then by the *restricted Lagrangian* corresponding to the orbit $G(v)$ the smooth function

$$L \circ \chi : G/G_v \to \mathbb{R}$$

is meant in case of any $v \in T_eG - \{0_e\}$ by a slightly loose terminology in what follows.

Lemma 2.1. *Let $L : TG \to \mathbb{R}$ be a left-invariant Lagrangian over a connected Lie group, $(y^1, \dots, y^m)$ the canonical coordinate system of G defined by a base $(E_1, \dots, E_m)$ of T_eG and*

$$(x^1, \dots, x^m; \dot{x}^1, \dots, \dot{x}^m)$$

the induced natural coordinate system of the tangent bundle TG. Let $v \in T_eG - \{0_e\}$ have the coordinates $(0, \dots, 0; v^1, \dots, v^m)$. Then v is a critical point of the restricted Lagrangian $L \circ \chi$ if and only if

$$\sum_{j,k=1}^{m} c_{ij}^k v^j \frac{\partial \tilde{L}}{\partial \dot{x}^k} = 0 \qquad i = 1, \dots, m$$

holds where $c_{ij}^k, i, j, k = 1, \dots, m$ are the structural constants of $\boldsymbol{g}$ in the above base and $\tilde{L}(x^1, \dots, x^m; \dot{x}^1, \dots, \dot{x}^m)$ is the coordinate expression of the Lagrangian $L \lceil TU$.

Proof. If $Z \in \boldsymbol{g}$ and $\tilde{Z}$ is the corresponding infinitesimal generator of the adjoint action ad then its complete lift $\tilde{Z}^c$ is the corresponding infinitesimal generator of the adjoint action Ad. Let $\bar{Z}^* : G/G_v \to T(G/G_v)$ be the infinitesimal generator of the canonical action of G on G/G_v corresponding to $Z \in \boldsymbol{g}$. Considering that χ is equivariant between the canonical action of G on G/G_v and its adjoint action Ad the above coordinate expression of $\tilde{Z}^c$ yields the following:

$$\iota_{Z^*} d(L \circ \chi) = \iota_{\tilde{Z}^c(V)} \mathrm{d}L = \tilde{Z}^c(V)L = \sum_{k=1}^{m} \sum_{i,j=1}^{m} c_{ij}^k \zeta^i v^j \frac{\partial \tilde{L}}{\partial \dot{x}^k}\bigg|_V = 0$$

where $Z = \sum_{i=1}^{m} \zeta^i E_i \in \boldsymbol{g}$ and the fact was used that $x^i(v) = 0, i = 1, \dots, m$ holds for $v \in T_eG$. Since the tangent space $T_vG(v)$ of the orbit $G(v)$ is spanned by the vectors $\tilde{Z}^c(v), Z \in \boldsymbol{g}$ the assertion of the lemma follows. □

Proposition 2.2. *Let G be a compact Lie group, $L : TG \to \mathbb{R}$ a left-invariant Lagrangian which is a first integral of its Lagrangian field X and $(y^1, \ldots, y^m)$ the canonical coordinate system of the Lie group defined by a base $(E_1, \ldots, E_m)$ of T_eG which is orthonormal with respect to an Ad-invariant Euclidean inner product of T_eG. Also let*

$$(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m)$$

be the natural coordinate system of TG induced by the above canonical one. Then $v \in T_eG - \{0_e\}$, is a geodesic vector if and only if

$$\frac{\partial \tilde{L}}{\partial x^i}(0, \ldots, 0; v^1, \ldots, v^m) = 0 \qquad i = 1, \ldots, m$$

holds where $\tilde{L}(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m)$ is the coordinate expression of $L \lceil TU$ and also $v = \sum_{i=1}^m v^i E_i$.

Proof. Since L is left-invariant, $\bar{Z}^c L = 0$ holds. However, then by the coordinate expression of $\bar{Z}^c \lceil TU$ the following is obtained:

$$\begin{aligned}\bar{Z}^c \lceil TU &= \sum_{k=1}^m \left\{ \zeta^k \frac{\partial \tilde{L}}{\partial x^k} + \sum_{i,j=1}^m c_{ij}^k \zeta^i \dot{x}^j \frac{\partial \tilde{L}}{\partial \dot{x}^k} \right\} \\ &= \sum_{k=1}^m \zeta^k \left\{ \frac{\partial \tilde{L}}{\partial x^k} + \sum_{i,j=1}^m c_{ij}^k \dot{x}^i \frac{\partial \tilde{L}}{\partial \dot{x}^k} \right\} = 0.\end{aligned}$$

Since the above equality holds for any $Z \in \boldsymbol{g}$, it yields the following system of equalities:

$$\frac{\partial \tilde{L}}{\partial x^k} + \sum_{i,j=1}^m c_{ij}^k \dot{x}^i \frac{\partial \tilde{L}}{\partial \dot{x}^j} = 0 \qquad k = 1, \ldots, m.$$

On the other hand, as L is a first integral of X, $XL = 0$ also holds. Therefore, the coordinate expression of X and the preceding system of equalities yield the following:

$$\begin{aligned}XL\lceil TU &= \sum_{l=1}^m \left\{ \dot{x}^l \frac{\partial \tilde{L}}{\partial x^l} + \xi^l(x^1, \ldots, x^l; \dot{x}^1, \ldots, \dot{x}^m) \frac{\partial \tilde{L}}{\partial \dot{x}^l} \right\} \\ &= \sum_{l=1}^m \left\{ \dot{x}^l \left(- \sum_{i,j=1}^m c_{ij}^l \dot{x}^i \frac{\partial \tilde{L}}{\partial \dot{x}^j} \right) + \xi^l(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m) \frac{\partial \tilde{L}}{\partial \dot{x}^l} \right\} \\ &= - \sum_{i,j,l=1}^m c_{ij}^l \dot{x}^i \dot{x}^l \frac{\partial \tilde{L}}{\partial \dot{x}^j} + \sum_{l=1}^m \xi^l(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m) \frac{\partial \tilde{L}}{\partial \dot{x}^l} = 0.\end{aligned}$$

Since the base $(E_1, \ldots, E_m)$ is orthonormal with respect to an Ad-invariant Euclidean interior product, the structural constants of $\boldsymbol{g}$ with respect to the above base satisfy the conditions $c_{ij}^l = -c_{lj}^i$, $i, j, l = 1, \ldots, m$. Therefore, the above expression simplifies as follows:

$$XL\lceil TU = \sum_{j=1}^m \xi^j(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m) \frac{\partial \tilde{L}}{\partial \dot{x}^j} = 0.$$

However, then the coordinate expression of $XL = 0$ reduces to the following:

$$XL\lceil TU = \sum_{l=1}^m \dot{x}^l \frac{\partial \tilde{L}}{\partial x^l} = 0.$$

Since XL vanishes, its vertical differential $\mathrm{d}_v(XL)$ vanishes too and therefore the following holds:

$$\mathrm{d}_v(XL)\lceil TU = \sum \frac{\partial(\tilde{XL})}{\partial \dot{x}^k}\,\mathrm{d}x^k = \sum_{k=1}^m \left\{ \frac{\partial \tilde{L}}{\partial x^k} + \sum_{l=1}^m \dot{x}^l \frac{\partial^2 \tilde{L}}{\partial \dot{x}^k \partial x^l} \right\} \mathrm{d}x^k = 0.$$

On the other hand, the Euler–Lagrange equations for L in the above induced natural coordinates of TG yield the following (see, e.g., [B, pp 24–6])

$$\sum_{i=1}^m \frac{\partial^2 \tilde{L}}{\partial \dot{x}^i \partial \dot{x}^l} \xi^i(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m) = \frac{\partial}{\partial x^l} - \sum_{i=1}^m \frac{\partial^2 \tilde{L}}{\partial x^i \partial \dot{x}^l} \dot{x}^i \qquad l = 1, \ldots, m.$$

Consider now the geodesic γ of L defined by $v = \dot{\gamma}(0)$ where $v = \sum_{i=1}^m v^i E_i \in T_eG$. According to proposition 1.4 the geodesic γ is stationary if and only if

$$\xi^i(0, \ldots, 0; v^1, \ldots, v^m) = 0 \qquad i = 1, \ldots, m$$

is valid. However, then the above coordinate expression of the Euler–Lagrange equation yields that γ is stationary if and only if

$$\frac{\partial \tilde{L}}{\partial x^l} - \sum_{i=1}^m \frac{\partial^2 \tilde{L}}{\partial x^i \partial \dot{x}^l} v^i = 0 \qquad l = 1, \ldots, m$$

holds, since the Lagrangian L is assumed to be regular. On the other hand, by the calculations above $XL = 0$ implies the following:

$$\frac{\partial \tilde{L}}{\partial x^l} + \sum_{k=1}^m \frac{\partial^2 \tilde{L}}{\partial \dot{x}^l \partial x^k} \dot{x}^k = 0 \qquad l = 1, \ldots, m.$$

By addition of the corresponding equations of the last two systems the following is obtained:

$$\frac{\partial \tilde{L}}{\partial x^l}(0, \ldots, 0; v^1, \ldots, v^m) = 0 \qquad l = 1, \ldots, m$$

is valid if and only if the geodesic γ is stationary. □

Theorem 2.3. *Let G be a compact Lie group and $L : TG \to \mathbb{R}$ a left-invariant Lagrangian which is a first integral of its Lagrangian field X. Then $v \in T_eG - \{0_e\}$ is a geodesic vector if and only if v is the image under the equivariant immersion $\chi : G/G_v \to G(v) \subset T_eG$ of a critical point of the restricted Lagrangian $L \circ \chi : G/G_v \to \mathbb{R}$.*

Proof. Consider the canonical coordinate system $(y^1, \ldots, y^m)$ of G defined by a base $(E_1, \ldots, E_m)$ of T_eG which is orthonormal with respect to an Ad-invariant Euclidean inner product of T_eG. Let no

$$(x^1, \ldots, x^m; \dot{x}^1, \ldots, \dot{x}^m)$$

be the natural coordinate system of TG induced by the above canonical one. If $v = \sum_{i=1}^m v^i E_i$ then according to proposition 2.2 the geodesic γ defined by $\dot{\gamma}(0) = v$ is stationary if and only if

$$\frac{\partial \tilde{L}}{\partial x^l}(0, \ldots, 0; v^1, \ldots, v^m) = 0 \qquad l = 1, \ldots, m$$

is valid. However, according to a system of equalities in the proof of proposition 2.2 the above equalities are valid if and only if the following hold:

$$\sum_{i,j=1}^{m} c_{ij}^{l} v^{i} \frac{\partial \tilde{L}}{\partial \dot{x}^{j}} = 0 \qquad l = 1, \ldots, m.$$

However, considering the identities $c_{ij}^{k} = -c_{kj}^{i}$, $i, j, k = 1, \ldots, m$ the assertion of the theorem now follows by *lemma 2.1*. □

3. The existence of stationary geodesics

As the energy function of a Riemannian manifold is homogeneous of degree two the following simple lemma which concerns homogeneous Lagrangians applies to this case as well. Consequently, the theorem below also yields a generalizations of a results concerning left-invariant Riemannian metrics.

Lemma 3.1. *If a regular Lagrangian $L : TM \to \mathbb{R}$ is homogeneous of degree k, where $k \neq 1$, then L is a first integral of its Lagrangian field X.*

Proof. In fact, the Euler–Lagrange equation takes now the following form:

$$\iota_X \, \mathrm{d}\mathrm{d}_v L = \mathrm{d}(L - AL) = \mathrm{d}(L - kL) = (1-k)\,\mathrm{d}L$$

and therefore $(1-k)\iota_X \, \mathrm{d}L = 0$ follows. However, since $k \neq 1$ holds, $XL = 0$ is obtained. □

In the following theorem two geodesics are considered different if their images are different.

Theorem 3.2. *Let G be a compact connected Lie group and $L : TG \to \mathbb{R}$ a left-invariant Lagrangian which is a first integral of its Lagrangian field. Then L has at least one stationary geodesic. If, in particular, G is also semi-simple and of* rank $\geqslant 2$ *then L has infinitely many stationary geodesics.*

Proof. Fix a $v \in T_eG - \{0_e\}$ then the quotient manifold G/G_v is compact and therefore the restricted Lagrangian

$$L \circ \chi : G/G_v \to \mathbb{R}$$

has at least one critical point which is mapped by χ to geodesic vectors w according to theorem 2.3.

If G is semi-simple then the negative of the Killing form $K : \boldsymbol{g} \times \boldsymbol{g} \to \mathbb{R}$ yields a Euclidean inner product $\langle\,,\rangle$ on $\boldsymbol{g}$. As two geodesics are considered different if they have different images, two geodesic vectors $w, w' \in \boldsymbol{g}$ yield different stationary geodesics issuing from the identity element if and only if there is no $\lambda \in \mathbb{R}$ with $w' = \lambda w$. Consequently, it suffices to show that the unit sphere S of $\langle\,,\rangle$ contains infinitely many geodesic vectors. Since now any homogeneous manifold G/G_v, $v \in S$ is compact, it contains at least one critical point of $L \circ \chi$. Therefore, by theorem 2.3 it is enough to see that S includes infinitely many adjoint orbits. However, since

$$\operatorname{codim} G(v) = \dim \boldsymbol{g} - \dim G(v) = \dim \boldsymbol{g} - (\dim G - \dim G_v) = \dim G_v \geqslant \operatorname{rank} G \geqslant 2$$

holds, the number of adjoint orbits included in the sphere S cannot be finite. □

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